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Arithmetic Billiard Paths Revisited: Escaping from a Rectangular Room

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***Abstract*— In this work, we consider a problem where a robot can move in a straight line inside a 2D rectangular room with integer lengths until it hits any part of the wall of the room. If the robot hits any part of a wall other than the corners or any point of an opening, then the robot bounces off the wall and follows a new direction in another straight line following the laws of symmetric reflection. The robot needs to escape through an opening on the wall that has a minimum length of one unit. The robot can only escape through the opening if it reaches any point of the opening with a non-zero angle.**

We present an efficient algorithm for which the robot is guaranteed to find the opening if there is any or declare that there is none. We prove that the algorithm works if and only if the sides of the rectangle are co-prime. As a by-product of our main result, we also provide some interesting results related to the coverage of the interior of the rectangle when the robot follows similar algorithms to escape from the rectangular room.

I. Introduction

Given a 2D rectangular room and a robot with no sensors, the goal is to find a possible opening on the walls, which is not known to the robot. The robot is assumed to have negligible dimensions and the length of the opening is at least 1 unit. When the robot hits any point of the opening with a positive angle then it can move through the opening and escape from the room. We call it an "Escape Room" problem.

The robot does not have any sensors and only uses one bit of memory, similar to the robot used in some recent works [1], [2], [3]. Since there are no environmental sensors, the robot continues to move along a straight line until it hits a wall other than any corner or any point of the opening. Once it hits a wall, it reorients itself and starts to move in a new direction. This is referred to as a *bounce*. In our case, the robot always makes a *symmetric bounce*, where the angle of incidence is always equal to the angle of reflection, see Figure 1. The path that the robot follows while bouncing off the sides of the rectangle is called a *billiard path*. Note that if the width and height of the rectangle are integers, the path is called an *arithmetic billiard path*.

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From a practical perspective this particular problem is of significant importance. Consider a 2D closed rectangular room and there is one important feature (Example, an opening) in the perimeter of the room which is continuous and has a minimum length of unit size. The room is not accessible by human, yet that important features need to be discovered. For that we could send a robot inside the rectangular room to discover that feature. Here we consider a very inexpensive robot (may be just a small ball with a motor) which just bounces back like a ball once it hits any wall. If the robot has no sensor then it will just follow the path of least resistance and uses the symmetric bounce algorithm and continue moving in the new direction after bounce. In our work we provide an algorithm which for a particular class of rectangles (sides which are co-prime) the robot is guaranteed to find the feature or declare that there is none.

The robot stops if it hits any corner. Though, in some cases, to derive our main result we hypothetically allow the robot to start again when it hits a corner. However, our final result is derived strictly with the condition that the robot stops when it hits any corner.

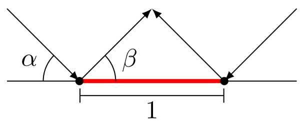

Fig. 1: An example of a symmetric bounce on the wall made by the robot. α is the angle of incidence, β is the angle of reflection. Here, $\alpha = \beta = \frac{\pi}{4}$. The discovered portion of the wall is indicated by thick red line segment. The hit points are indicated by a black dot.

Let a and b be the side lengths of a given rectangular room such that $a, b \in \mathbb{Z}_+, a \leq b$. The robot starts its movement from a point on the vertical side which are at a distance $y_0 \in [0, a]$ from the nearest corner, with an angle of $\alpha \in \left[0, \frac{\pi}{2}\right]$ counterclockwise from the vertical axis.

The opening cannot span across two walls. That means if any point of the opening is touching a corner, then that point is one end of the opening, and the other end would be at least 1 distance away.

If the robot hits any point on a wall with a non-zero angle, and doesn't escape then it is guaranteed that there is no opening that touches that hitting point. We call all those hitting points visited points. If the distance between two visited points is at most 1, then it is guaranteed that there is no opening between these two visited points. That means the robot doesn't need to visit those portions of the wall. Those lengths of the wall which are guaranteed not to have openings are called "discovered" points. From this, it follows that if a visited point is less than one unit distance away from any corner then the length between the corner and that visited point is discovered.

II. Organization of the Paper

First, we discuss the related work in Section III. Then, in Section IV we establish the framework to study the case where the starting angle $\alpha = \frac{\pi}{4}$. We then discuss some important results in Section V which form the basis for our main result. Then, in Section VI, we discuss and theoretically prove our main results when the lengths of the rectangular room are co-prime. As a by-product of our main result, we provide some additional interesting observations in Section VII. Finally, we offer insights on the possible directions and future steps in Section VIII.

III. Related Work

Multiple works have been focused on the *coverage path planning* problems which determine whether all the points in the free space are covered by the given robot [4]. There are several applications for the studies on coverage path planning problems; for example, see [5], [6]. Some applications are lawn mowing and milling [5] where the path has a thickness, and the goal is to find the shortest path of the robot such that it covers the whole interior of the given polygon.

Several studies have tackled the problem of navigating a bouncing robot and considered it a combinatorial problem. Sinai [7] analyzed the dynamic system which is formed by the motion of a material point and proved that such a system is ergodic; in contrast, Boldrighini et al. [8] looked at the computational aspects of bouncing by viewing the system as a billiard ball bouncing inside a polygon and following the rules of reflection. Some studies have explored random bouncing within a polygon, as opposed to following the laws of physics [3]. Tokarsky [9] examined the possible shots on a polygonal billiard table to reach a target point, finding that not all points can be reached unless the bouncing rule is random. Aronov et al. [10] studied the diffuse and specular paths of the rays emanating from a point assuming that an edge of the polygon is a mirror. Recently, Perucca et. al. [11] studied the properties of arithmetic billiard paths.

IV. Preliminaries

We give some definitions and important preliminary lemmas in this section. Throughout this paper, the robot always starts with an angle $\frac{\pi}{4}$ with the sides.

Lemma 1. *Consider the robot starts its motion from a point on the boundary of the rectangular room with the angle of $\frac{\pi}{4}$. If the robot hits the same point twice, then the path repeats.*

Proof. From the laws of symmetric reflection and the fact that rectangle sides are either parallel or perpendicular to each other, it follows that for every bounce on the wall, the angle of incidence and the angle of reflection is $\frac{\pi}{4}$. Thus, if the robot hits a point twice then it will always use the angle $\frac{\pi}{4}$ to move away from that point. From the same point, if the robot uses the same direction then it arrives at the same set of points going further. Thus, the path repeats. □

Definition 1. *Within the rectangular room, if the robot bounces on any vertical side, it changes the direction of its horizontal movement either from left to right or right to left. Every such change of horizontal direction is called a* ***switch****. The complete movement from one vertical side to another vertical side is called a* ***traversal****.*

It follows from Definition 1 that starting from a vertical side if the robot makes k switches, it makes $k+1$ traversals.

Lemma 2. *Consider the robot starting its motion from a corner of the rectangular room with angle $\frac{\pi}{4}$. If the sides of the rectangle are co-prime, the robot hits another corner after $a+b-2$ number of bounces. Out of that total $a-1$ number of bounces will be on the horizontal sides and the total of $b-1$ number of bounces will be on the vertical sides.*

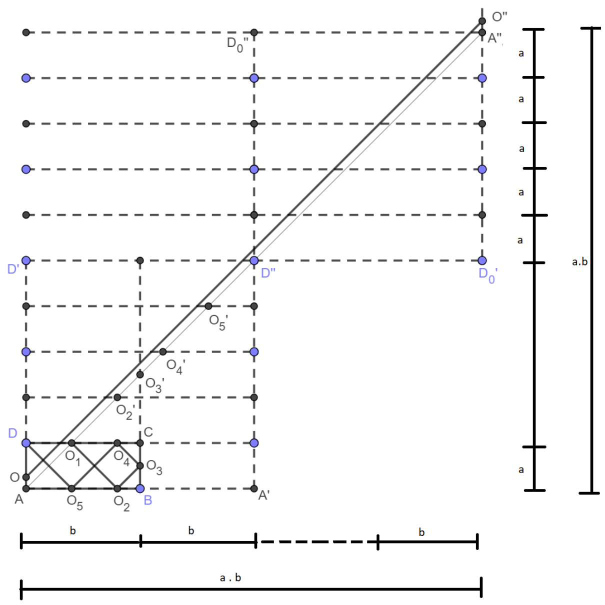

Fig. 2: Arithmetic Billiard Bounce

Proof. The path followed by the reflection on a wall is equivalent to the straight path taken by the robot going through the wall on the rectangle that is created by the reflection of the actual rectangle about the wall (unfolding the rectangle about the wall). Refer to Figure 2. Here, the rectangle is reflected about the vertical side a times, and each of these reflected rectangles is again reflected about the horizontal side b times. This way all the combined unfolded rectangles, will create a square of side length $(a \cdot b)$.

If the robot starts from one corner of that square with side length $(a \cdot b)$ at an angle $\frac{\pi}{4}$, then it will hit the corner of that square which is diagonally opposite to the starting corner. The path that goes from one corner to its diagonally opposite corner in the square of length $(a \cdot b)$, is equivalent to the path that the robot follows in the original rectangle, where it starts from one corner at an angle $\frac{\pi}{4}$ with the sides and after bouncing on the walls it lands on another corner of the rectangle. In this case, the robot crosses the vertical lines $b-1$ times or equivalently reflects $b-1$ times on the horizontal (longer) side and similarly $a-1$ times on the vertical (shorter) side. Thus, there are a total $(a+b-2)$ number of bounces without counting the initial and starting corner points. Because those sides a and b are co-prime, a square cannot be formed smaller than the sides $a \cdot b$. That means, in this case, the robot first hits any corner only after the $(a+b-2)$ bounce.

Now we prove that when the robot first hits a corner then it will not be the corner from where it started. If the robot hits the same corner then it has to reflect an even number of times both horizontally and vertically, but because a and b are co-prime, one of them must be odd. Thus, when the robot first hits any corner it cannot be the corner from where it started. □

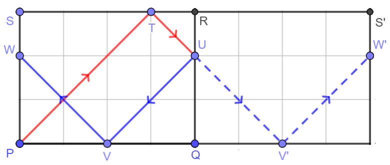

Fig. 3: Perpetual Bounce

We can also demonstrate this bouncing process in an alternative way. If the robot starts from one corner. then at every bounce, the robot moves a distance away horizontally from the corner. Thus, if the side b was a multiple of a then the robot would hit one corner, which is not possible as a and b are co-prime. There would be some remainder when b is divided by a. Hence, the robot will hit the other vertical side and then bounce back. This bouncing on the vertical side is equivalent to continuing in a straight line through the wall inside the unfolded rectangle, which was created reflecting the rectangle about the wall. Refer to Figure 3. Thus, if it finishes total k traversals then the robot would have gone through the k unfolded rectangles of size b, which is again equivalent to 1 traversal of a rectangle of horizontal side $k \cdot b$. If $k = a$, then there is no remainder as then $k \cdot b$ is divided by a. Hence, the robot will hit the corner of the rectangle of side length $a \cdot b$ after a traversals or $a-1$ switch. Thus, there will be a total of $a-1$ bounces on the vertical (shorter) sides, before it hits the corner. Because a and b are co-prime, if $k < a$, then there will be a remainder if $(k \cdot b)$ is divided by a. Hence the robot will not hit the corner before a traversals or equivalently $a-1$ bounce on the vertical side. Similarly, there are $a \cdot b / a = b$ numbers of a sized vertical shifts from the original corner in the $a \cdot b$ side rectangle. Thus, there are total $b-1$ bounces on the horizontal (longer) sides.

If the robot starts from a side and after several bounces and traversals come back to the same vertical sides from where it started, then the robot needs to make traversals an even number of times. Moreover, if the robot needs to move back exactly to the corner from where it started, then it needs to bounce off horizontal walls an odd number of times.

If a is even then it makes traversals an even number of times. However, because a and b are co-prime, then if a is even then b cannot be even and $b-1$ cannot be odd; thus, it cannot bounce an odd number of times before it hits the corner. Thus, when the robot hits the corner the first time it cannot go back to the starting corner when the sides are co-prime.

V. Main Results

From the previous discussions, it follows that if the robot is allowed to continue after it hits a corner with $\frac{\pi}{4}$ angle, it will trace back the path and eventually arrive at the original starting point. Hence, we derive the following observation.

Observation 1. *Starting from a corner of a rectangular room, if we deviate from our original condition and allow the robot to continue after it hits a corner with $\frac{\pi}{4}$ angle, it will come back to the original starting point after $2 \cdot (b-1)$ number of horizontal bounces and $2 \cdot (a-1)$ number of vertical bounces counting from the start. Thus, if we include the bounces on the corner that the robot first hits, it will return to the original corner after $2 \cdot (a+b-2)+1 = 2 \cdot (a+b)-3$ number of total bounces.*

From the above discussion, it follows that if the numbers a and b are not co-prime, then using the perpetual reflected rectangles (unfolded rectangles about the reflection walls) a square would form with sides $\left(\frac{a \cdot b}{g}\right)$. Thus, starting from a corner the robot will hit a corner at $\left(\frac{a+b}{g}\right)-2$ bounce and come back to the initial corner after $2\left(\frac{a+b}{g}\right)-3$ bounce.

Lemma 3. *If the robot starts from a vertical side and then hits the other vertical side at a distance r from a corner, the maximum distance between any two closest visited points on any side is* $\max\{2r, 2(a-r)\}$.

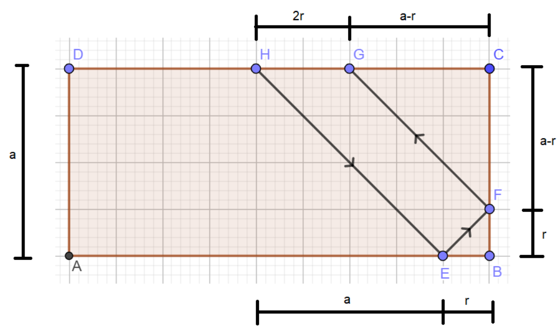

Fig. 4: Distance between visited points

Proof. Consider the robot hits a vertical side coming from a horizontal side, we denote that horizontal side by AB and the other horizontal side which is opposite to AB as CD; we denote the vertical side as BC. We denote the corner where BC and AB meet as B, and the corner where BC and CD meet as C.

Let the hitting point on BC be at a distance r from the corner B. It follows that the last point on AB before hitting the vertical side BC is also r distance away from the corner B. Hence, the last point H the robot hit on CD must be $r + a$ distance away from the corner C. Refer to Figure 4.

After hitting the vertical side BC the robot will move to the side CD and will hit a point G at $a - r$ distance from the corner. Thus, the distance between the points H and G is $(a+r)-(a-r) = 2r$, and the distance of the point G from the previous visited point N on CD is $2a - 2r = 2(a - r)$. Therefore, the distance from any corner to the nearest visited point is less than the $\max\{2r, 2(a-r)\}$. This process is also similar to the horizontal sides as well. For any horizontal side, the maximum distance between the two nearest visited points after one switch is $\max\{2r, 2(a - r)\}$. Furthermore, the maximum distance from any corner to the adjacent visited points is $\max\{r, (a - r)\}$. It also follows that the maximum distance between any two visited points on the vertical side is $\max\{r, (a-r)\}$. Combining these, the maximum distance between the two nearest visited points after one switch is $\max\{2r, 2(a - r)\}$. □

Lemma 4. *Consider $x, y \in \mathbb{Z}_+$ which are co-prime and $x < y$. Let R be the collection of all the remainders of $p \cdot y \mod x, \forall p \in \{1, \dots, x\}$. Then, $R = \{0, \dots, x - 1\}$.*

Proof. If $p = x$ then, $(p \cdot y \mod x) = 0$. Here, x, y are co-primes. So, $\forall p \in \{1, \dots, x - 1\}, \quad x > (p \cdot y \mod x) > 0$. The number of members of the collection R is x, where one of the member of R is 0 and the other members can range from 1 to $x - 1$. In order to guarantee that the members of R are all the integers from 0 to $x-1$, it must be shown that the $x - 1$ number of non-zero members of the collection R are unique. This in turn follows that $R = \{0, \dots, x - 1\}$.

If the non-zero members of R are not unique then there exists at least two members within R which are same. Without loss of generality, let $(q_1 \cdot y \mod x) = (q_2 \cdot y \mod x) = r$, where $q_1, q_2 \in \{1, \dots, x - 1\}$, which implies $q_1 \cdot y = n_1 \cdot x + r$ and $q_2 \cdot y = n_2 \cdot x + r$, where $n_1, n_2 \in \mathbb{N}$. Which again implies $(q_1 - q_2) \cdot y = (n_1 - n_2) \cdot x$. Here, it is clear that if $q_1 > q_2$, then $n_1 > n_2$. We define, $q := (q_1-q_2)$, which means $q \in \{1, \dots, x - 1\}$. Thus, in that case $q \cdot y$ is divisible by x, i.e, $q \cdot y \mod x = 0$, which is impossible. Thus, the members of R are unique and they are integers ranging from 0 to $x - 1$. Thus, $R = \{0, \dots, x - 1\}$. □

Thus, if the robot made the traversal a times then $y_a = (a \cdot k) \mod a = 0$. Thus, after a number of traversals there will be no remainder and the robot will hit a corner. Moreover, the robot will encounter different remainders for every switch of every hit on the vertical side as per Lemma 4. Thus, for every hit on the vertical sides the distances from the nearest corners will be unique before encountering a corner. In other words, the starting point for every traversal will be unique.

Lemma 5. *If the robot starts from a corner, when it hits a corner, the maximum distance between the two adjacent visited points is* 2.

Proof. If the robot performs k number of traversals, then it is equivalent to one traversal within the rectangle with vertical side a and horizontal side $(k \cdot b)$.

When a robot bounces on the horizontal sides, in each bounce it moves a distance horizontally. Suppose the robot starts from a corner of a vertical side L and moves to the opposite vertical side R, which is $(k \cdot b)$ away from the side L. If the robot does not hit any corner of R, then the last bounce of the robot on a horizontal side just before it hits R will be some distance away from R. Let that distance be r, which is given by $r = (k \cdot b) \mod a$.

Let the robot start from a corner and reach another corner with a number of traversals. This is equivalent to 1 traversal in the rectangles with vertical side a and horizontal side $(k \cdot b)$, for each $k \in \{1, \dots, a\}$, which will generate a number of different remainders or $a - 1$ number of unique non-zero remainders, as per Lemma 4. In every traversal, the maximum distance between two adjacent points will decrease by twice the changes in the remainder.

Hence, at the end when the robot hits the corner, the difference between the encountered remainders will be one; thus the maximum distance between two adjacent visited points will be $2 \cdot 1 = 2$. □

Lemma 6. *If y_0 is not an integer, then the robot can not hit any corner.*

Proof. Refer to Figure 2. Considering that one corner of the rectangle is the origin and its adjacent sides are the axes. If the robot starts from the origin with an angle $\frac{\pi}{4}$ with the sides, then within the $2D$ space it will always touch the points with integer coordinates. If it starts from any integer distance away from the corner, then it will also always touch the points with integer coordinates; however, if it starts from a fractional distance away from the corner and travels with an angle $\frac{\pi}{4}$, then it can never touch the points with integer coordinates. Since the corners are the points with integer coordinates, starting from a fractional distance away from the corner with an angle $\frac{\pi}{4}$, the robot will never touch any corners.

This can also be demonstrated analytically: if the robot starts at y_0 distance away from the corner and y_k is the distance from its nearest corner after k traversals, then it is given by $y_k = k \cdot b + y_0 - a \cdot n$, where n is an integer and given by $\left\lfloor \frac{b+y_0}{a} \right\rfloor$. Here, k, b, n, a are all integers. Thus, if y_0 is not an integer, then y_k can never be an integer, and no corner can be reached. □

Lemma 7. *Suppose that the robot starts its motion from a point on one of the vertical sides at a distance $y_0 > 0$ from the corner on the boundary of a rectangular room with the angle of $\frac{\pi}{4}$. If y_0 is not an integer, then it comes back to the starting position after $2 \cdot (a + b) - 1$ bounces, out of which $2 \cdot b$ bounces are on the horizontal edges and $2 \cdot a - 1$ number of bounces are on the vertical edges.*

Proof. If we refer to Figure 2, then the path starting from the corner and the path starting from the y_0 are parallel. The path from the corner starts and returns to point A, and when the path is plotted on the reflected rectangles (unfolded rectangles) the path goes straight to point A''. The path starting from y_0 starts at point O and when the path is plotted on the reflected rectangles the path goes straight to the point $0''$. Now, $AA''||OO''$. Thus, $\overline{A''O''} = \overline{AO} = y_0$. That means the path comes back to the original point which is on the vertical side and at a distance y_0 from the corner. However, in this case, the path OO'' crosses the horizontal lines two more times and the vertical lines one more time than the path AA'' and also misses the corner point D'. This path makes a total of $2(b-1)+2 = 2b$ number of crossings or bounces on the horizontal (longer) sides and $2(a-1)+1 = 2a-1$ crossings or bounces on the vertical (shorter) sides. Thus, it will take a total of $2(a+b)-1$ bounces before the robot returns to its original position and will not hit any corner during these bounces. □

VI. Discovering the Whole Boundary

In the worst case scenario, the robot needs to discover the entire boundary before it can escape through the opening or declare there is no opening. In this section, we discuss an algorithm that guarantees the whole boundary of the rectangle is discovered by the robot. Unless otherwise specified we assume the sides of the rectangular room are co-prime.

Theorem 1. *If the robot starts from a point on a vertical side of the rectangular room at a distance $\frac{1}{2}$ from the corner, with an angle $\frac{\pi}{4}$, the robot discovers the whole perimeter of the room.*

Proof. If a robot starts from a point on a vertical side at a distance $\frac{1}{2}$ from the corner, with angle $\frac{\pi}{4}$, the robot after a number of traversals will reach the non-starting vertical side at a point which is $\frac{1}{2}$ distance away from the corner.

At that time the maximum distance between the two adjacent visited points is 2. Thus, it will complete a traversal back to the original position and for that, it will create the maximum distance between two adjacent points as $2 \cdot \frac{1}{2} = 1$.

Hence, each visited point will cover $\frac{1}{2}$ length on each side and two adjacent visited points will combine the total $\frac{1}{2} + \frac{1}{2} = 1$ length, which is the maximum distance between them. It follows that the adjacent visited points from each corner will be separated by a distance of $\frac{1}{2} < 1$. Thus, the robot will discover the whole perimeter. □

From the above results, it follows that if a robot starts from a point on a vertical side at a distance $\frac{1}{2}$ from the corner, with an angle of $\frac{\pi}{4}$ then the robot will come back to the initial point after $2(a+b)-1$ bounces. This ensures that the maximum distance between two adjacent visited points is 1 only if a, b are co-prime, otherwise, the robot will return to the original position, repeating the loop earlier than the required $2(a+b)-1$ bounces. This causes the maximum distance between two adjacent visited points to be greater than 1, meaning the whole perimeter will not be discovered.

Thus, we then create our main result by modifying the previous theorem.

Theorem 2. *If the robot starts from a point on a vertical side of the rectangular room, at a distance of $\frac{1}{2}$ from any corner, with an angle of $\frac{\pi}{4}$, then the robot discovers the whole perimeter if and only if the sides are integer and co-prime.*

Theorem 3. *If the robot starts with the angle of $\frac{\pi}{4}$, from a point at a distance $\frac{1}{2}$ on any vertical side from any corner of a rectangular room, it travels a distance of $2\sqrt{2} \cdot a \cdot b$ when it finishes discovering the whole perimeter.*

Proof. Refer to Figure 2, which shows the total distance covered by the robot when it starts $\frac{1}{2}$ distance away from a corner and returns to the starting point. This traversal is equivalent to the distance when the robot starts from a corner hits another corner and returns to the starting point. This is equivalent to the distance covered by the robot starting from a corner and hitting another corner of a rectangle of vertical side a and horizontal side $(2 \cdot a \cdot b)$.

Here, when the robot moves from one point on the perimeter to the next point on the perimeter (called a step) in the rectangle of size $(2 \cdot a \cdot b) \times a$ the robot moves horizontally and vertically a distances in each step. That means in every step the robot covers a total of $\sqrt{2} \cdot a$ distance. Horizontally it covers $(2 \cdot a \cdot b)$ distance when it hits a corner and it covers a distance in each step. So it finishes $(2 \cdot b)$ steps when it hits a corner. Thus, the total distance the robot covers from starting and back to the starting point is $(2 \cdot b) \cdot a\sqrt{2} = 2\sqrt{2} \cdot a \cdot b$. □

VII. Further Observations

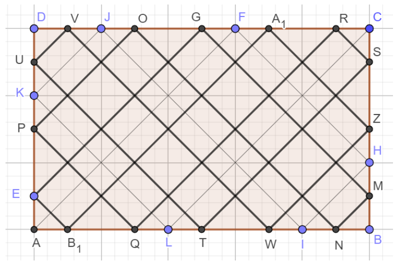

Fig. 5: Simulating bounce of a robot inside a rectangle

Observation 2. *If the opening length is 2 then the robot can start from a corner and by the time the robot hits the first corner the whole perimeter of the rectangle is discovered.*

Proof. This observation follows directly from Lemma 5. Refer to the thin lines in Figure 5. These thin lines represent the path where the robot starts from a corner A and it ends at another corner C. In this case, as per Lemma 5, the distances between the adjacent visited points are 2, and the distances between any corner and its closest visited points are 1. So, the whole perimeter will be visited. □

Observation 3. *Consider that the rectangle is divided into unit squares. If the robot starts from one corner, by the time it hits the first corner it passes through all the squares.*

Proof. The path between two consecutive points on the longer sides is called a step. It follows that in every step, the robot passes through a number of squares. We know that the robot hits the first corner after b number of steps on the horizontal side. Thus, when it hits the first corner it passes through a total of $a \cdot b$ squares. After every switch, the robot follows a new set of squares. Thus, it will pass through a total of $a \cdot b$ squares when it hits the first corner. There are $a \cdot b$ unique squares in the rectangle of sides a and b. Thus, the robot will pass through all the unit squares by the time it hits the first corner. Refer to Figure 5. □

Observation 4. *Consider that the rectangle is divided into unit squares and the opening length is $\frac{1}{2}$. By the time the robot discovers the whole boundary, it passes through all the squares four times.*

Proof. In Theorem 2 we have demonstrated that the whole boundary is discovered when the robot starts from a smaller side at $\frac{1}{2}$ distance away from a corner. We call this algorithm the "half-start algorithm". When the robot starts from one corner, we call the algorithm the "corner algorithm". We refer to Figure 2 and from there it is clear that half of the total path covered by the "half-start algorithm" is the same as the path covered by the "corner algorithm". Also, the paths covered by these two algorithms are parallel to each other when the rectangle is unfolded in accordance with the bounces. We have demonstrated that the robot passes through all the unit squares following the corner-algorithm. Thus, the robot should also pass through all the unit squares following the half-start algorithms within half of its total path. However, in the corner algorithm, the robot covers the full length of the diagonals of a unit square and only the half length of the diagonal using the half-start algorithm, yet because their length is the same, the half-start algorithm must pass through each unit square twice within half of the time it takes to discover the whole boundary. The half-start algorithm never repeats before it finishes the discovery of the perimeter. Thus, each half of the path covered by the robot, using the half-start algorithm, will pass each square two times uniquely. This concludes that the robot must pass through each square total of four times when it finishes discovering the whole boundary. □

Observation 5. *By the time the robot discovers the whole boundary of the given rectangle, the maximum radius of the biggest circle that can be fit inside the rectangle through which the robot never passed through, is $\frac{1}{2\sqrt{2}}$.*

Proof. From the previous observations (4) it follows that the path of our proposed algorithm passes through each of the midpoints of each square. Those paths in a unit square will create a square of length $\frac{1}{2}\sqrt{2}$. So the diameter of the biggest circle that can be fit inside that square would be the same as its side, i.e. $\frac{1}{\sqrt{2}}$. So its radius would be $\frac{1}{2\sqrt{2}}$. □

VIII. Future Work and Conclusion

In this work, we have studied minimalist robots which follow the billiard path movement in a rectangular environment with an opening of length one unit. We have proposed an algorithm in which it is guaranteed that the robot will either escape through the opening or declare there is no opening of the rectangle when the width and height of the rectangle are co-prime. Following this, we have also found interesting observations mentioned in section VII.

This work is a pillar for ergodic dynamic system designs explained in [12], [13], [14], [15] for wild bodies, namely bouncing robots. More precisely, we are interested to know what path the robot would follow when the lengths of the room are not co-prime, or if the shape of the room is a parallelogram or any arbitrary polygon instead of a rectangle. Another interesting problem is to consider other types of strategies for bouncing, for example, the robot bounces off the obstacle not in a symmetric way. Another interesting question is to analyze the coverage of the inside of the room instead of the boundary.

References

[1] L. Bobadilla, F. Martinez, E. Gobst, K. Gossman, and S. M. LaValle, "Controlling wild mobile robots using virtual gates and discrete transitions," in *2012 American Control Conference (ACC)*. IEEE, 2012, pp. 743–749.

[2] D. Gierl, L. Bobadilla, O. Sanhcez, and S. LaValle, "Stochastic modeling, control, and evaluation of wild bodies," in *IEEE International Conference on Robotics and Automation*, 2014.

[3] A. Q. Nilles, Y. Ren, I. Becerra, and S. M. LaValle, "A visibility-based approach to computing non-deterministic bouncing strategies," *The International Journal of Robotics Research*, vol. 40, no. 10-11, pp. 1196–1211, 2021.

[4] H. Choset, "Coverage for robotics–a survey of recent results," *Annals of mathematics and artificial intelligence*, vol. 31, pp. 113–126, 2001.

[5] E. M. Arkin, S. P. Fekete, and J. S. Mitchell, "Approximation algorithms for lawn mowing and milling," *Computational Geometry*, vol. 17, no. 1-2, pp. 25–50, 2000.

[6] S. Ramamoorthy, R. Rajagopal, Q. Ruan, and L. Wenzel, "Low-discrepancy curves and efficient coverage of space," in *Algorithmic Foundation of Robotics VII: Selected Contributions of the Seventh International Workshop on the Algorithmic Foundations of Robotics*. Springer, 2008, pp. 203–218.

[7] Y. G. Sinai, "Dynamical systems with elastic reflections," *Russian Mathematical Surveys*, vol. 25, no. 2, p. 137, 1970.

[8] C. Boldrighini, M. Keane, and F. Marchetti, "Billiards in polygons," *The Annals of Probability*, pp. 532–540, 1978.

[9] G. W. Tokarsky, "Polygonal rooms not illuminable from every point," *The American Mathematical Monthly*, vol. 102, no. 10, pp. 867–879, 1995.

[10] B. Aronov, A. R. Davis, T. K. Dey, S. P. Pal, and D. C. Prasad, "Visibility with one reflection," *Discrete and Computational Geometry*, vol. 19, no. 4, pp. 553–574, 1998.

[11] A. Perucca, J. R. De Sousa, and S. Tronto, "Arithmetic billiards," *Recreational Mathematics Magazine*, vol. 9, no. 16, pp. 43–54, 2019.

[12] D. A. Shell, C. V. Jones, and M. J. Matarić, "Ergodic dynamics by design: A route to predictable multi-robot systems," in *Multi-Robot Systems. From Swarms to Intelligent Automata Volume III: Proceedings from the 2005 International Workshop on Multi-Robot Systems*. Springer, 2005, pp. 291–297.

[13] L. M. Miller, Y. Silverman, M. A. MacIver, and T. D. Murphey, "Ergodic exploration of distributed information," *IEEE Transactions on Robotics*, vol. 32, no. 1, pp. 36–52, 2015.

[14] A. T. Taylor, T. A. Berrueta, and T. D. Murphey, "Active learning in robotics: A review of control principles," *Mechatronics*, vol. 77, p. 102576, 2021.

[15] O. Çağırıcı, Y. Bahoo, and S. M. LaValle, "Bouncing robots in rectilinear polygons," in *2022 26th International Conference on Methods and Models in Automation and Robotics (MMAR)*. IEEE, 2022, pp. 193–198.